Effect of Discrete Time Sampling from Stochastic Differential Equation on the Consistency of Parameter Estimate

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Abstract

The order of consistency of parameter estimate of Itô type stochastic differential equation is investigated for the case when observations are obtained at finite number of time points. Three types of estimates are examined, the exact maximum likelihood estimate, the maximum likelihood estimate based on Euler-Maruyama approximation and a bootstrap estimate. The result shows that the exact maximum likekihood estimate is consistent for a wide variety of choices of sampling interval and a careful choice of the sampling interval gives us the highest order of consistency but the Euler-Maruyama estimate is only consistent for a limited choice of the sampling interval. The bootstrap estimate behaves similarly to the exact maximum likelihood estimate and applicable for wider type of stochatic differential equatiion than Ornstein-Uhlenbeck or Longnomal process which are mainly investigated in this paper.

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1 Introduction

In this paper, we will investigate the effect of discrete time sampling on the estimation of the parameters $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_p)^T$ of a stochastic process $\{X_t, t \geq 0\}$, which satisfies a Markovian type stochastic differential equation,

$$dX_t = a(X_t, t; \boldsymbol{\theta})dt + b(X_t, t; \boldsymbol{\theta}) \cdot dB_t. \tag{1.1}$$

Here $\{B_t, t \geq 0\}$ is a standard Brownian motion and the product $b(X_t, t; \boldsymbol{\theta}) \cdot dB_t$ is defined as a limit in probability of

$$\sum_{i=1}^{n} b(X_{t_{i-1}}, t_{i-1}; \boldsymbol{\theta})(B_{t_i} - B_{t_{i-1}})$$

as the partition $\{[t_{i-1}, t_i)\}_{i=1}^n$ of dt becomes finer and finer. Although this type of product, Itô product, is commonly used in practice from Hydrology to Option Pricing theory(Black and Sholes, 1973, Lo, 1986, 1988), there could be many other ways of defining the product. An example is Stratonovich's product. From the view point of statistical modeling, different definition of the product yields different understandings of phenomenon which stochastic differential equation describes(Sethi and Lehoczky, 1981). We don't go further into this issue in this paper, but only direct the reader's attention to the practical importance of the problem which definition of the product is appropriate for each phenomenon.

The main concern in this paper is the order of consistency of parameter estimate when observation is only obtained at finite number of time points, $0 \leq t_0 < t_1 < t_2 < \cdots < t_n$. For simplicity, we assume that all time intervals $[t_i, t_{i-1}), i = 1, \cdots, n$ have the same length Δ_n which may depend on the number of observations $n+1$. It becomes clear that the order of consistency of parameter estimate heavily depends on the choice of Δ_n and on the type of estimation procedure. Even the consistency of the estimate does not hold true unless Δ_n is carefully chosen so as to satisfy appropriate conditions required for each estimate. We will clarify such a machinery for each type of estimates, the maximum likelihood estimate based on the exact solution, which we call *exact maximum likelihood* estimate, the maximum likelihood estimate based on Euler-Maruyama approximation, which we call *Euler-Maruyama* estimate, and the bootstrap estimate. We restrict our attention into two types of stochastic differential equations or the processes, Ornstein-Uhlenbeck process,

$$dX_t = -\beta X_t dt + \sigma dB_t \tag{1.2}$$

or Lognormal process,

$$dX_t = \mu X_t dt + \sigma X_t \cdot dB_t. \tag{1.3}$$

One of reasons why we restrict our attention into such two special processes is that the exact solution of the equation is known for those two processes so that it is easy to compare different estimates exactly. We believe that our results can be generalized for any other process, but we leave it for future investigation.

Our result shows that the exact maximum likelihood estimate is consistent for a wide variety of choices of Δ_n and the order of consistency is the highest in most cases. In case of Ornstein-Uhlenbeck process, the highest order of consistency $\sqrt{n}$ is achieved for the case when $\beta > 0$ as far as the sequence Δ_n is bounded and bounded away from 0. On the other hand, if $\beta < 0$ there is no bound for the order of consistency. That is, the faster divergence of Δ_n, the higher order of consistency. This is because the randomness due to Brownian motion is negligible relative to the explosive drift $-\beta X_t dt$ in (1.2). In case of Lognormal process, the faster divergence of Δ_n, the higher order of consistency of $\hat{\mu}_{MLE}$, irrespective of the value of μ. In either case, the exact maximum likelihood estimate looses its consistency if Δ_n converges too fast to zero. For the Euler-Maruyama estimate, the range of Δ_n which yields the consistency is narrower than that for the exact maximum likelihood estimate and the order of consistency is also lower. For example, the order $\sqrt{n}$ is never attained by any choice of Δ_n when $\beta > 0$. On the other hand, the bootstrap estimate is quite promising. The same range of Δ_n as that for the exact maximum likelihood estimate yields the consistency and the same order of consistency is achieved, as far as the re-sampling size is large enough. Although there is no need to use such a bootstrap estimate when the exact solution of the equation is known, the result suggests a power of bootstrap type estimate even when the exact solution is unknown.

It may be dangerous to conclude something concrete from such limited two case studies, but we believe that our result describes an aspect of an effect of discrete time sampling on the order of consistency of parameter estimate. We leave any generalization of our result for future investigation.

2 Order of Consistency of Parameter Estimate

We will investigate the consistency of three types of estimates one by one. The results are summarized in the last section. We always assume that $\sigma > 0$.

2.1 Exact Maximum Likelihood Estimate

In this subsection, we will investigate the order of consistency of the exact maximum likelihood estimate separately for Ornstein-Uhlenbeck process and Lognormal process.

2.1.1 Ornstein-Uhlenbeck Process

The exact solution of the stochastic differential equation (1.2) is well known that

$$X_t = \mathrm{e}^{-\beta(t-t_0)}\{X_{t_0} + \sigma \int_{t_0}^{t} \exp(\beta u) dB_u\}.$$

If we rewrite it as

$$X_{t_i} - X_{t_{i-1}} \mathrm{e}^{-\beta\Delta_n} = \mathrm{e}^{-\beta(t_i - t_0)} \sigma \int_{t_{i-1}}^{t_i} \exp(\beta u) dB_u,$$

then, from the Markov property of the process the conditional log likelihood for observations $x_{t_0}, x_{t_1}, \cdots, x_{t_n}$, given x_{t_0} is given by

$$\begin{aligned} l(\boldsymbol{\theta}) &= \log f_{\boldsymbol{\theta}}(x_{t_n}, \cdots, x_{t_1} | x_{t_0}) \\ &= -\frac{n}{2} \log(2\pi\gamma) - \frac{1}{2\gamma} \sum_{i=1}^{n} (x_{t_i} - \exp(-\beta\Delta_n) x_{t_{i-1}})^2. \end{aligned}$$

For convenience, a parameter transform from (β, σ) to (β, γ) is employed,

$$\gamma = \begin{cases} \sigma^2 \exp(2\beta t_0)\{1 - \exp(-2\beta\Delta_n)\}/2\beta & \beta \neq 0 \\ \sigma^2 \Delta_n & \text{otherwise.} \end{cases}$$

In this paper we always call such a conditional likelihood $l(\boldsymbol{\theta})$ simply the likelihood. The maximum likelihood estimate of $\boldsymbol{\theta} = (\beta, \gamma)^T$ is then given by

$$\hat{\beta}_{MLE} = -\frac{1}{\Delta_n} \log \frac{\sum_{i=1}^{n} x_{t_{i-1}} x_{t_i}}{\sum_{i=1}^{n} x_{t_{i-1}}^2}$$

and

$$\hat{\gamma}_{MLE} = \frac{1}{n}\sum_{i=1}^{n}\left(x_{t_i} - \exp\left(-\hat{\beta}_{MLE}\Delta_n\right)x_{t_{i-1}}\right)^2$$
$$= \frac{1}{n}\sum_{i=1}^{n}\left(x_{t_i} - \frac{\sum_{j=1}^{n} x_{t_{j-1}}x_{t_j}}{\sum_{j=1}^{n} x_{t_{j-1}}^2}x_{t_{i-1}}\right)^2 .$$

The exact maximum likelihood estimate of σ^2 is thus

$$\hat{\sigma}^2_{MLE} = \hat{\gamma}_{MLE}\cdot 2\hat{\beta}_{MLE}\exp\left(-2\hat{\beta}_{MLE}\, t_0\right)\Big/\left(1-\exp\left(-2\hat{\beta}_{MLE}\Delta_n\right)\right). \quad (2.4)$$

We need the following lemmas to evaluate the moments of such estimates. Without loss of generality, we may assume that $t_0 = 0$. Otherwise it is enough to replace σ^2 by $\sigma^2\exp(2\beta t_0)$ in the following lemmas. We hereafter use the notation $d_n = \exp(-\beta\Delta_n)$ for simplicity.

Lemma 1

If $\beta = 0$,

$$\mathrm{E}\left(\sum_{i=1}^{n} X_{t_{i-1}}^2 \,\middle|\, X_{t_0} = x_{t_0}\right) = \left(x_{t_0}^2 + \frac{n(n-1)}{2}\Delta_n\right)\sigma^2$$

and

$$\mathrm{E}\left(\left(\sum_{i=1}^{n} X_{t_{i-1}}^2\right)^2 \,\middle|\, X_{t_0} = x_{t_0}\right) = \frac{n(n-1)\sigma^4}{2}\left\{\frac{7}{12}n^2\Delta_n^2 + \left(\frac{7}{3}x_{t_0}^2\Delta_n - \frac{13}{12}\Delta_n^2\right)n + x_{t_0}^4 - \frac{4}{3}x_{t_0}^2\Delta_n + \frac{1}{12}\Delta_n^2\right\}.$$

Otherwise,

$$\mathrm{E}\left(\sum_{i=1}^{n} X_{t_{i-1}}^2 \,\middle|\, X_{t_0} = x_{t_0}\right) = \left\{\frac{n}{2\beta} + \left(x_{t_0}^2 - \frac{1}{2\beta}\right)\left(\frac{1-d_n^n}{1-d_n}\right)\right\}\sigma^2$$

and

$$\mathrm{E}\left(\left(\sum_{i=1}^{n} X_{t_{i-1}}^2\right)^2 \,\middle|\, X_{t_0} = x_{t_0}\right)$$

$$
\begin{aligned}
= \;& \sigma^4 \left\{ \left(x_{t_0}^4 - \frac{3}{\beta} x_{t_0}^2 + \frac{3}{4\beta^2} \right) \frac{1-d_n^{2n}}{1-d_n^2} + \left(\frac{3}{\beta} x_{t_0}^2 - \frac{3}{2\beta^2} \right) \frac{1-d_n^n}{1-d_n} \right. \\
& + \frac{n(n-2)}{4\beta^2} + \left(2x_{t_0}^4 - \frac{7}{\beta} x_{t_0}^2 + \frac{1}{\beta^2} \right) \frac{d_n - d_n^n}{(1-d_n)^2} \\
& - \frac{d_n^2 - d_n^{n+1}}{2\beta^2 (1-d_n)^2} - \left(2x_{t_0}^4 - \frac{6}{\beta} x_{t_0}^2 + \frac{1}{2\beta^2} \right) \frac{d_n^2 - d_n^{2n}}{(1-d_n)^2 (1+d_n)} \\
& + \left(\frac{5}{\beta} x_{t_0}^2 - \frac{5}{2\beta^2} \right) \left(\frac{d_n - n d_n^n + (n+1) d_n^{n+1}}{(1-d_n)^2} \right) \\
& \left. + \left(\frac{x_{t_0}^2}{\beta} - \frac{(1-r)}{2\beta^2} \right) \left(\frac{n-1}{1-d_n} \right) \right\}.
\end{aligned}
$$

Since the proof of Lemma 1 is straightforward, we omit the proof.

Lemma 2

$$
\mathrm{E}\left(\sum_{i=1}^{n} X_{t_{i-1}} \left(X_{t_i} - d_n X_{t_{i-1}} \right) \middle| X_{t_0} = x_{t_0} \right) = 0
$$

and

$$
\begin{aligned}
& \mathrm{E}\left(\left(\sum_{i=1}^{n} X_{t_{i-1}} (X_{t_i} - d_n X_{t_{i-1}}) \right)^2 \middle| X_{t_0} = x_{t_0} \right) \\
& = \begin{cases} \left\{ x_{t_0}^2 (1-\Delta_n) n \Delta_n + \frac{n(n-1)}{2} \Delta_n^2 \right\} \sigma^4 & \text{if } \beta = 0, \\ \left[\frac{x_{t_0}^2}{2\beta} (1-d_n^{2n}) + \frac{1}{8\beta^2} \left\{ n(1-d_n) - (1-d_n^{2n}) \right\} \right] \sigma^4 & \text{otherwise.} \end{cases}
\end{aligned}
$$

Proof

Define $W_i = \int_{t_{i-1}}^{t_i} \exp(\beta u) dB_u, i = 1, \cdots, n$ with $W_0 = 0$. Then we can write

$$
\sum_{i=1}^{n} X_{t_{i-1}} (X_{t_i} - d_n X_{t_{i-1}}) = \sum_{i=1}^{n} \exp(-\beta(t_{i-1} + t_i))(X_{t_0} + \sum_{j=1}^{i} W_j) W_i.
$$

The lemma for the case when $\beta = 0$ is trivial, and we prove it in other cases. Without loss of generality, we assume that $\sigma = 1$. Since W_i's are independent and normally distributed random variables with mean 0 and variance

$$
\delta_i = \frac{1}{2\beta} (\exp(2\beta t_i) - \exp(2\beta t_{i-1})), \; i = 1, \cdots, n,
$$

we can write

$$\mathrm{E}\left(\sum_{i=1}^{n} X_{t_{i-1}}(X_{t_i} - d_n X_{t_{i-1}})\middle| X_{t_0} = x_{t_0}\right) = \mathrm{E}(W^T C W)$$

and

$$\mathrm{E}\left\{\left(\sum_{i=1}^{n} X_{t_{i-1}}(X_{t_i} - d_n X_{t_{i-1}})\right)^2 \middle| X_{t_0} = x_{t_0}\right\}$$
$$= x_{t_0}^2 \mathrm{E}\left(\sum_{i=1}^{n} \exp(-\beta(t_{i-1}+t_i))W_i\right)^2 + \mathrm{E}\left(\sum_{i=1}^{n} \exp(-\beta(t_{i-1}+t_i))\sum_{j=i}^{i-1} W_j W_i\right)$$
$$= \frac{x_{t_0}^2}{2\beta}(1 - d_n^{2n}) + \mathrm{E}(W^T C W)^2.$$

Here we used the notations of the vector of random variables $W = (W_1, \cdots, W_n)^T$ and the matrix

$$C = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ c_1 & 0 & \cdots & \cdots & \cdots & 0 \\ c_2 & c_2 & 0 & \cdots & \cdots & 0 \\ c_3 & c_3 & c_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_{n-1} & c_{n-1} & \cdots & c_{n-1} & 0 \end{bmatrix}$$

where $c_i = \exp(-\beta(t_{i-1} + t_i))$. To evaluate the moments of $W^T C W$, it is convenient to symmetrize C into $A = C + C^T$. Then, there exits a nonsingular matrix F such that

$$F^T A F = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \quad \text{and} \quad F^T \Sigma^{-1} F = I,$$

where $\Sigma = \mathrm{diag}(\delta_1, \cdots, \delta_n)$, and $\lambda_1, \cdots, \lambda_n$ are the roots of

$$\det(A - \lambda \Sigma^{-1}) = 0.$$

Define a vector of random variables $Y = (Y_1, \cdots, Y_n)^T = F^{-1} W$, then we have

$$W^T A W = 2 W^T C W = \sum_{i=1}^{n} \lambda_i Y_i^2.$$

Since $Y_1, \cdots, Y_n$ are independent and normally distributed random variables with mean 0 and variance 1, we have

$$\mathrm{E}(W^T C W) = \frac{1}{2}\sum_{i=1}^{n} \lambda_i$$

and

$$\mathrm{E}(W^T C W)^2 = \frac{1}{4}\sum_{i=1}^{n} \lambda_i^2.$$

If we note that the diagonal elements of $A\Sigma$ are all zeros, it follows that

$$\sum_{i=1}^{n} \lambda_i = \mathrm{tr}(A\Sigma) = 0$$

and

$$\sum_{i=1}^{n} \lambda_i^2 = \mathrm{tr}(A\Sigma)^2 = 2\sum_{k=1}^{n}\sum_{i=k+1}^{n} c_i^2 d_k d_i = \frac{n}{2\beta^2}(1-d_n) - \frac{1}{2\beta^2}(1-d_n^{2n}).$$

This completes the proof of Lemma 2.

To evaluate the order of consistency of $\hat{\beta}_{MLE}$ and of $\hat{\gamma}_{MLE}$ or $\hat{\sigma}^2_{MLE}$, it is convenient to introduce new variables,

$$\eta_n = \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}^2$$

and

$$\xi_n = \frac{1}{nd_n}\sum_{i=1}^{n} X_{t_{i-1}}(X_{t_i} - d_n X_{t_{i-1}}).$$

We can then write

$$\hat{\beta}_{MLE} - \beta = -\frac{1}{\Delta_n}\log\left(1 + \frac{\xi_n}{\eta_n}\right), \tag{2.5}$$

and

$$\begin{aligned} \hat{\gamma}_{MLE} - \gamma &= \frac{1}{n}\sum_{i=1}^{n}\left(x_{t_i} - d_n x_{t_{i-1}} - d_n \frac{\xi_n}{\eta_n} x_{t_{i-1}}\right)^2 - \gamma \\ &= \frac{1}{n}\sum_{i=1}^{n}(x_{t_i} - d_n x_{t_{i-1}})^2 - \gamma - d_n^2 \frac{\xi_n^2}{\eta_n}. \end{aligned} \tag{2.6}$$

To know the order of consistency of $\hat{\beta}_{MLE}$ it is necessary to find two normalizing constants a_n and b_n by which η_n/a_n and ξ_n/b_n have non-degenerate limiting distributions. Such constants can be easily found from the evaluation of moments in Lemma 1 and Lemma 2 since the underlying process is driven by a Brownian motion. The results are summarized in Table 1 and Table 2. As is easily seen, the constant a_n or b_n takes significantly different values according to both the value of parameters and the initial value x_{t_0} of the process.

Table 1. Normalizing constant a_n for η_n to have a non-degenerate limiting distribution.

		$\beta = 0$		
Δ_n	$\beta > 0$	$x_{t_0} = 0$	$x_{t_0} \neq 0$	$\beta < 0$
$n\Delta_n \to \infty$	1	$n\Delta_n$	$n\Delta_n$	$d_n^{2(n-1)}/n$
$n\Delta_n$ bounded	$(n\Delta_n)^{-\frac{1}{2}}$	$(n\Delta_n)^{\frac{1}{2}}$	1	$(n\Delta_n)^{-\frac{1}{2}}$

Table 2. Normalizing constant b_n for ξ_n to have a non-degenerate limiting distribution.

		$\beta = 0$		
Δ_n	$\beta > 0$	$x_{t_0} = 0$	$x_{t_0} \neq 0$	$\beta < 0$
$\Delta_n \to \infty$	$(\sqrt{n}d_n)^{-1}$	Δ_n	$(n/\Delta_n)^{-\frac{1}{2}}$	d_n^{n-1}/n
Δ_n bounded	$(n/\Delta_n)^{-\frac{1}{2}}$	Δ_n	$(n/\Delta_n)^{-\frac{1}{2}}$	$(n/\Delta_n)^{-\frac{1}{2}}$

We say that the order of consistency of an estimate $\hat{\theta}$ is c_n if the mean squared error $\mathrm{E}(\hat{\theta} - \theta)^2$ is exactly of the order of $1/c_n^2$ in its magnitude for any θ. To evaluate the order of consistency of an estimate we need further evaluation of the order of the ratio b_n/a_n which is summarized in Table 3. Such conditions for Δ_n are denoted in the table by D_1 to D_4:

D_1: Δ_n diverges to infinity.

D_2: Δ_n is bounded and bounded away from 0.

D_3: Δ_n converges to zero but $n\Delta$ diverges to infinity.

D_4: $n\Delta_n$ is bounded.

Table 3. Ratio of the normalizing constants: b_n/a_n.

Δ_n	$\beta > 0$	$\beta = 0$		$\beta < 0$
		$x_{t_0} = 0$	$x_{t_0} \neq 0$	
D_1	$(\sqrt{n}d_n)^{-1}$	$1/n$	$n^{-\frac{3}{2}}\Delta_n^{-\frac{1}{2}}$	$d_n^{-(n-1)}$
$D_2 \vee D_3$	$(n/\Delta_n)^{-\frac{1}{2}}$	$1/n$	$n^{-\frac{3}{2}}\Delta_n^{-\frac{1}{2}}$	$(n\Delta_n)^{\frac{1}{2}}d_n^{-2(n-1)}$
D_4	Δ_n	$(n/\Delta_n)^{-\frac{1}{2}}$	$(n/\Delta_n)^{-\frac{1}{2}}$	Δ_n

Noting that $\hat{\beta}_{MLE} - \beta$ is of the order of $\frac{1}{\Delta_n}\log(1 + b_n/a_n)$ in probability as is seen from (2.6) we have the following theorem.

Theorem 1.

The maximum likelihood estimate $\hat{\beta}_{MLE}$ of the parameter β of Ornstein-Uhlenbeck process is consistent only in the following cases.

Case 1. $\beta > 0$

If the condition D_1 is satisfied, then $\hat{\beta}_{MLE}$ is consistent as far as $\alpha_n = \Delta_n/\log(1 + 1/(\sqrt{n}d_n))$ diverges to infinity and if so the order of consistency is α_n. It is also consistent under the conditions D_2 or D_3 and the order of consistency is $(n\Delta_n)^{\frac{1}{2}}$.

Case 2. $\beta = 0$

$\hat{\beta}_{MLE}$ is consistent except for the case of D_4. The order of consistency is $n\Delta_n$ or $(n\Delta_n)^{\frac{3}{2}}$ according to x_{t_0} being zero or not.

Case 3. $\beta < 0$

$\hat{\beta}_{MLE}$ is consistent except for the case of D_4. The order of consistency is $\Delta_n d_n^{n-1}$ for the case of D_1 or D_2, and $(\Delta_n/n)^{\frac{1}{2}}d_n^{2(n-1)}$ for the case of D_3.

Proof

Most of the results directly follow from Table 3. We only give a part of the proof. For example, in Case 3, if Δ_n diverges to infinity then the ratio b_n/a_n is of the order of $d_n^{-(n-1)}$ which converges to zero. Therefore $\frac{1}{\Delta_n}\log(1+b_n/a_n)$ is of the order of $d_n^{-(n-1)}/\Delta_n$. If Δ_n is bounded but $n\Delta_n$ diverges to infinity, then the ratio b_n/a_n is of the order of $(n\Delta_n)^{\frac{1}{2}}d_n^{-2(n-1)}$ which goes to zero, so that $\frac{1}{\Delta_n}\log(1+b_n/a_n)$ is of the order of $(n/\Delta_n)^{\frac{1}{2}}d_n^{-2(n-1)}$. Finally, if the condition D_4 is satisfied, $\frac{1}{\Delta_n}\log(1+\Delta_n)$ never converges to zero, so that $\hat{\beta}_{MLE}$ is inconsistent.

For the consistency of $\hat{\sigma}^2_{MLE}$, we need first the following proposition which assures the consistency of $\hat{\gamma}_{MLE}$.

Proposition 1.

The maximum likelihood estimate $\hat{\gamma}_{MLE}$ of γ of Ornstein-Uhlenbeck process is always consistent with the order of consistency $\sqrt{n}$.

Proof

In (2.6), $\frac{1}{n}\sum_{i=1}^{n}(x_{ti}-d_n x_{t_{i-1}})^2-\gamma$ is of the order of $1/\sqrt{n}$ in probability since $X_{t_i}-d_n X_{t_{i-1}}, i=1,\cdots,n$ are independent and normally distributed random variables with mean 0 and variance γ. On the other hand, the last term $d_n^2\xi_n^2/\eta_n$ is of the order of $(d_n^2/(1-d_n^2))(b_n^2/a_n)$ in probability which is always less than $1/\sqrt{n}$ in its magnitude. This complete the proof of Proposition 1.

On the contrary to the consistency condition for $\hat{\gamma}_{MLE}$, that for $\hat{\sigma}^2_{MLE}$ is not so simple. The behavior of the estimate $\hat{\sigma}^2_{MLE}$ is affected by that of $\hat{\beta}_{MLE}$ through a link between those estimates shown in (2.4). From the following theorem we can see the effect of $\hat{\beta}_{MLE}$ to the estimate $\hat{\sigma}^2$.

Theorem 2.

The maximum likelihood estimate $\hat{\sigma}^2_{MLE}$ of the parameter σ^2 of Ornstein-Uhlenbeck process is consistent only in the following cases.

Case 1. $\beta > 0$

Except for the case of D_4, the estimate $\hat{\sigma}^2_{MLE}$ is consistent as far as α_n diverges to infinity, where α_n is the sequence defined in Theorem 1. In case of D_1, the order of consistency is $\sqrt{n}\Delta_n d_n$ if $\sqrt{n}d_n$ diverges to infinity and it is $\min(\sqrt{n},\alpha_n)$ otherwise. In case of D_2 or D_3, the order is $\sqrt{n}$ or $\sqrt{n\Delta_n}$ respectively.

Case 2. $\beta < 0$

Except for the case of D_4, the estimate $\hat{\sigma}^2_{MLE}$ is consistent. The order of consistency is $\min(\sqrt{n}, d_n^{n-1})$ or $\min(\sqrt{n}, (\Delta_n/n)^{\frac{1}{2}} d_n^{2(n-1)})$ according to the case D_2 or D_3.

Case 3. $\beta = 0, t_0 \neq 0$
Except for the case of D_4, the estimate $\hat{\sigma}^2_{MLE}$ is consistent. The order of consistency is $\sqrt{n}$ if the condition D_1 is satisfied. For the case of D_2 or D_3, the order of consistency varies according to x_{t_0} being zero or not. It is $\min(\sqrt{n}, (n\Delta_n)^{\frac{3}{2}})$ if $x_{t_0} = 0$, and $\min(\sqrt{n}, n\Delta_n)$ otherwise.

Case 4. $\beta = 0, t_0 = 0$
The estimate $\hat{\sigma}^2_{MLE}$ is always consistent and the order of consistency is $\sqrt{n}$.

Proof

When $\beta \neq 0$

$$\begin{aligned}
\frac{\hat{\sigma}^2_{MLE}}{\sigma^2} &= \frac{\hat{\gamma}_{MLE}}{\gamma} \exp(-2(\hat{\beta}_{MLE} - \beta)t_0) \frac{1 - d_n^2}{1 - d_n^2\left(1 + \frac{\xi_n}{\eta_n}\right)^2} \frac{\hat{\beta}_{MLE}}{\beta} \\
&= \frac{\hat{\gamma}_{MLE}}{\gamma} \left(1 + O_p\left(\frac{b_n}{a_n}\right)\right)^{2t_0/\Delta_n} \frac{1 - d_n^2}{1 - d_n^2\left(1 + O_p\left(\frac{b_n}{a_n}\right)\right)^2}.
\end{aligned}$$

If we remember the evaluations of the ratio b_n/a_n in Table 3 together with Proposition 1, then the desired results follow. The case $\beta = 0$ is special. Note that

$$\frac{\hat{\sigma}^2_{MLE}}{\sigma^2} = \frac{\hat{\gamma}_{MLE}}{\gamma} \frac{2\hat{\beta}_{MLE}\Delta_n \exp(-2\hat{\beta}_{MLE}t_0)}{1 - \exp(-2\hat{\beta}_{MLE}\Delta_n)},$$

in this case and

$$\frac{2\hat{\beta}_{MLE}\Delta_n}{1 - \exp(-2\hat{\beta}_{MLE}\Delta_n)} = \frac{2\log\left(1 + \frac{\xi_n}{\eta_n}\right)}{1 - \left(1 + \frac{\xi_n}{\eta_n}\right)^2}. \tag{2.7}$$

Since ξ_n/η_n always converges to zero in probability as is seen from Table 3, the right hand side of (2.7) is evaluated as $\left(1 + O_p\left(\frac{b_n}{a_n}\right)\right)^{-2}$. We have then

$$\frac{\hat{\sigma}^2_{MLE}}{\sigma^2} = \left(1 + O_p\left(\frac{1}{\sqrt{n}}\right)\right)\left(1 + O_p\left(\frac{b_n}{a_n}\right)\right)^{-2+\frac{2t_0}{\Delta_n}}.$$

from Proposition 1. The desired results for Case 3 and Case 4 follow from Table 3.

2.1.2 Lognormal Process

In case of lognormal process, the solution of the stochastic differential equation (1.3) is given by

$$X_t = X_{t_0} \exp\left\{(\mu - \frac{1}{2}\sigma^2)(t - t_0) + \sigma B_t\right\}.$$

By transforming observational variable $X_{t_i}, i = 0, \cdots, n$ into

$$Y_{t_i} = \log(X_{t_i}/X_{t_{i-1}}) = (\mu - \frac{\sigma^2}{2})\Delta_n + \sigma(B_{t_i} - B_{t_{i-1}}), \quad i = 1, \cdots, n,$$

we obtain the log likelihood

$$l(\boldsymbol{\theta}) = -\frac{n}{2}\log(2\pi\sigma^2\Delta_n) - \frac{1}{2\sigma^2\Delta_n}\sum_{i=1}^{n}(y_{t_i} - \alpha\Delta_n)^2$$

of $\boldsymbol{\theta} = (\alpha, \sigma^2)^T$ with $\alpha = \mu - \sigma^2/2$ when the transformed observations $y_{t_1}, \cdots, y_{t_n}$ are available. Since this log likelihood differers from the log likelihood based on the original observations $x_{t_i}, i = 0, \cdots, n$ only by a constant, the maximum likelihood estimate of $\boldsymbol{\theta}$ is unchanged by the log transformation of observations. Therefore

$$\hat{\alpha}_{MLE} = \frac{1}{n\Delta_n}\sum_{i=1}^{n} y_{t_i} = \frac{1}{n\Delta_n}\log\frac{x_{t_n}}{x_{t_0}}$$

and

$$\begin{aligned}\hat{\sigma}^2_{MLE} &= \frac{1}{n\Delta_n}\sum_{i=1}^{n}(y_{t_i} - \hat{\alpha}_{MLE}\Delta_n)^2 \\ &= \frac{1}{n\Delta_n}\sum_{i=1}^{n}\left(\log\frac{x_{t_i}}{x_{t_{i-1}}} - \frac{1}{n}\log\frac{x_{t_n}}{x_{t_0}}\right)^2.\end{aligned}$$

Since $Y_{t_i}, i = 1, \cdots, n$ are independent and normally distributed random variables with the mean $\alpha\Delta_n$ and the variance $\sigma^2\Delta_n$, $\hat{\sigma}^2_{MLE}$ has the mean $(1 - 1/n)\sigma^2$ and the variance $(2/n)\sigma^4$. Thus $\hat{\alpha}_{MLE}$ has the mean α and the variance $\sigma^2/(n\Delta_n)$. Therefore $\hat{\sigma}^2_{MLE}$ is always consistent with the order

$\sqrt{n}$ irrespective of the choice of Δ_n. On the other hand, $\hat{\mu}_{MLE} = \hat{\alpha}_{MLE} + (1/2)\hat{\sigma}^2_{MLE}$ has the mean $-(\sigma^2/2n)$ and the variance $\sigma^2/n\Delta_n + \sigma^4/2n$, so that it is consistent if and only if $n\Delta_n$ diverges to infinity. Summarizing these results, we have the following theorem.

Theorem 3.

The maximum likelihood estimate $\hat{\mu}_{MLE}$ of the parameter μ of Lognormal process is consistent except for the case of D_4 and the order of consistency is given by $\min(\sqrt{n}, (n\Delta_n)^{\frac{1}{2}})$. The maximum likelihood estimate $\hat{\sigma}^2_{MLE}$ is always consistent with the order of $\sqrt{n}$ irrespective of the choice of Δ_n.

2.2 Euler-Maruyama Estimate

2.2.1 Ornstein-Uhlenbeck process

Euler-Maruyama approximation is simply to approximate a stochastic differential equation by a stochastic difference equation. The differential equation (1.2) is approximated to

$$X_{t_i} - X_{t_{i-1}} = -\beta X_{t_{i-1}}(t_i - t_{i-1}) + \sigma(B_{t_i} - B_{t_{i-1}}) \,. \tag{2.8}$$

Regarded that the process $\{X_t\}$ satisfies this difference equation, we have the log likelihood of $\boldsymbol{\theta} = (\beta, \sigma^2)^T$ for given observations $x_{t_i}, i = 0, \cdots, n$,

$$l(\boldsymbol{\theta}) = -\frac{n}{2}\log(2\pi\sigma^2\Delta_n) - \frac{1}{2\sigma^2\Delta_n}\sum_{i=1}^{n}(x_{t_i} - (1-\beta\Delta_n)x_{t_{i-1}})^2 \,.$$

Strictly speaking, this is in fact the conditional log likelihood given x_{t_0}, but we call it simply the log likelihood in the same way as in Section 2.1. The maximum likelihood estimate then becomes

$$\hat{\beta}_{EU} = \frac{1}{\Delta_n}\left(1 - \frac{\sum_{i=1}^n x_{t_{i-1}}x_{t_i}}{\sum_{i=1}^n x^2_{t_{i-1}}}\right)$$

and

$$\hat{\sigma}^2_{EU} = \frac{1}{n\Delta_n}\sum_{i=1}^{n}\left(x_{t_i} - \frac{\sum_{j=1}^n x_{t_{j-1}}x_{t_j}}{\sum_{j=1}^n x^2_{t_{j-1}}}x_{t_{i-1}}\right)^2 .$$

If we note that

$$\hat{\beta}_{EU} - \beta = \frac{1}{\Delta_n}\left\{(1-\beta\Delta_n) - \exp(-\beta\Delta_n)\left(\frac{\zeta_n}{\eta_n}+1\right)\right\},$$

we have the following theorem from Table 3 in the same way as in the preceding section. In the following theorems we evaluate the behavior of the estimate under the assumption that the observation follows the differential equation (1.3), although the estimate is derived from the difference equation (2.8). It is important to distinguish the model used for estimation from that generates observations.

Theorem 4.

The Euler-Maruyama estimate $\hat{\beta}_{EU}$ of the parameter β of Ornstein-Uhlenbeck process is consistent in the following cases. Otherwise $\hat{\beta}_{EU}$ is inconsistent.

Case 1. $\beta > 0$
$\hat{\beta}$ is consistent only when the condition D_3 is satisfied and the order of consistency is $(n\Delta_n)^{\frac{1}{2}}$.

Case 2. $\beta = 0$
Except for the case of D_4, $\hat{\beta}_{EU}$ is consistent and the order of consistency is $n\Delta_n$ if x_{t_0} being zero and $(n\Delta_n)^{\frac{3}{2}}$ otherwise.

Case 3. $\beta < 0$
$\hat{\beta}_{EU}$ is consistent only when the condition D_3 is satisfied and the order of consistency is $\Delta_n d_n^{n-1}$.

For the consistency of $\hat{\sigma}^2_{EU}$, we have the following theorem.

Theorem 5.

The Euler-Maruyama estimate $\hat{\sigma}^2_{EU}$ of Ornstein-Uhlenbeck process is consistent in the following cases. Otherwise $\hat{\sigma}^2_{EU}$ is inconsistent.

Case 1. $\beta \neq 0$
If the condition D_3 or D_4 is satisfied, then $\hat{\sigma}^2_{EU}$ is consistent if $t_0 = 0$. The order of consistency is $\min(\sqrt{n}, 1/\Delta_n)$.

Case 2. $\beta = 0$
$\hat{\sigma}^2_{EU}$ is always consistent and the order of consistency is $\sqrt{n}$.

Proof

Note that $\hat{\sigma}^2_{EU} = \hat{\gamma}_{MLE}/\Delta_n$. If $\beta \neq 0$, then

$$\frac{\hat{\sigma}^2_{EU}}{\sigma^2} = \frac{\exp(2\beta t_0)(1 - \mathrm{e}^{-2\beta\Delta_n})}{2\beta\Delta_n}\frac{\hat{\gamma}_{MLE}}{\gamma}$$

Since $\hat{\gamma}_{MLE}$ is a consistent estimate of γ with the order $\sqrt{n}$ as is seen in Proposition 1, $\hat{\sigma}^2_{EU}$ is consistent if and only if Δ_n converges to zero and $t_0 = 0$. Otherwise a significant bias remains. If $\beta = 0$, then

$$\frac{\hat{\sigma}^2_{EU}}{\sigma^2} = \frac{\hat{\gamma}_{MLE}}{\gamma}$$

Therefore $\hat{\sigma}^2_{EU}$ is always consistent with the order $\sqrt{n}$ from Proposition 1.

2.2.2 Lognormal Process

Euler-Maruyama approximation to Lognormal process leads us to the difference equation,

$$X_{t_i} - X_{t_{i-1}} = \mu X_{t_{i-1}}(t_i - t_{i-1}) + X_{t_{i-1}}\sigma(B_{t_i} - B_{t_{i-1}}).$$

Regarded that $\{X_t\}$ satisfies this difference equation, the random variables

$$Y_{t_i} = \frac{X_{t_i} - X_{t_{i-1}}}{X_{t_{i-1}}}, \quad i = 0, \cdots, n$$

are independent and normally distributed with the mean $\mu\Delta_n$ and the variance $\sigma^2\Delta_n$. The log likelihood function then becomes

$$l(\boldsymbol{\theta}) = -\frac{n}{2}\log(2\pi\sigma^2\Delta_n) - \frac{1}{2\sigma^2\Delta_n}\sum_{i=1}^{n}(y_{t_i} - \mu\Delta_n)^2.$$

If we remember that the transformation from $\{X_{t_i}\}$ to $\{Y_{t_i}\}$ only causes a difference of likelihood functions by a constant, we have the maximum likelihood estimate of $\boldsymbol{\theta} = (\mu, \sigma^2)^T$ as

$$\hat{\mu}_{EU} = \frac{1}{n\Delta_n}\sum_{i=1}^{n} y_{t_i} = \frac{1}{n\Delta_n}\sum_{i=1}^{n}\frac{x_{t_i} - x_{t_{i-1}}}{x_{t_{i-1}}}$$

and

$$\hat{\sigma}^2_{EU} = \frac{1}{n\Delta_n}\sum_{i=1}^{n}(y_{t_i} - \hat{\mu}_{EU}\Delta_n)^2 = \frac{1}{n\Delta_n}\sum_{i=1}^{n}\left(\frac{x_{t_i} - x_{t_{i-1}}}{x_{t_{i-1}}} - \hat{\mu}_{EU}\Delta_n\right)^2 .$$

There is another way to derive the maximum likelihood estimate by using Euler-Maruyama approximation: first apply the log transformation $Y_t = \log(X_t)$ and next apply Euler-Maruyama approximation. Then the differential equation (1.3) becomes

$$dY_t = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dB_t \tag{2.9}$$

by the log transform and the Euler-Maruyama approximation yields

$$Y_{t_i} - Y_{t_{i-1}} = (\mu - \frac{1}{2}\sigma^2)(t_i - t_{i-1}) + \sigma(B_{t_i} - B_{t_{i-1}}) .$$

As a result this equation is not an approximation but an exact difference equation which $\{Y_t\}$ satisfies. Therefore, the maximum likelihood estimate based on such Y_{t_i}'s coincides with the exact maximum likelihood estimate $\hat{\mu}_{MLE}$ and $\hat{\sigma}^2_{MLE}$ before. This is a specific property of Lognormal process where Euler-Maruyama approximation causes no approximation error. The reason is that all coefficients of the differential equation for $\{Y_t\}$ in (2.9) being non-random. Thus in the following, we will evaluate the behavior of the previously defined $\hat{\mu}_{EU}$ and $\hat{\sigma}^2_{EU}$.

To evaluate the consistency of $\hat{\mu}_{EU}$ and $\hat{\sigma}^2_{EU}$, it is enough to note that $\log(X_{t_i}/X_{t_{i-1}})$, $i = 1, \cdots, n$ are independent and normally distributed random variables with mean $(\mu - \sigma^2/2)\Delta_n$ and variance $\sigma^2\Delta_n$ under the assumption that $\{X_t\}$ satisfies the differential equation (1.3). Thus

$$\mathrm{E}(\hat{\mu}_{EU} - \mu) = \frac{1}{\Delta_n}(\exp(\mu\Delta_n) - 1) - \mu$$

and

$$\mathrm{Var}(\hat{\mu}_{EU}) = \frac{1}{n\Delta_n}\exp(2\mu\Delta_n)(\exp(\sigma^2\Delta_n) - 1) .$$

Also

$$\mathrm{E}(\hat{\sigma}^2_{EU} - \sigma^2) = \left(1 - \frac{1}{n}\right) \frac{1}{\Delta_n} \exp(2\mu\Delta_n)\{\exp(\sigma^2\Delta_n) - 1\} - \sigma^2$$

and

$$\begin{aligned}\mathrm{Var}(\hat{\sigma}^2_{EU}) &= \frac{1}{n^4}\Big\{ n(n-1)^2 R_1 - (n^3 - 2n^2 + 3n - 4)R_2 \\ &\quad +8(n-1)^2 R_3 + 2(n^4 - 4n^3 + 9n^2 - 6n - 4)R_4 \\ &\quad -(n^4 - 2n^3 + 3n^2 + 20 - 24)R_5 \Big\},\end{aligned}$$

where

$$\begin{aligned}R_1 &= \exp(2(2\mu + 3\sigma^2)\Delta_n) - 4\exp(3(\mu + 2\sigma^2)\Delta_n) \\ &\quad +6\exp((2\mu + \sigma^2)\Delta_n) - 4\exp(\mu\Delta_n) + 1 \\ R_2 &= \exp(2(2\mu + \sigma^2)\Delta_n) - 4\exp((3\mu + \sigma^2)\Delta_n) + 2\exp((2\mu + \sigma^2)\Delta_n) \\ &\quad +4\exp(2\mu\Delta_n) - 4\exp(\mu\Delta_n) + 1 \\ R_3 &= \exp((4\mu + 3\sigma^2)\Delta_n) - \exp(3(\mu + \sigma^2)\Delta_n) - 3\exp((3\mu + \sigma^2)\Delta_n) \\ &\quad +3\exp((2\mu + \sigma^2)\Delta_n) + 3\exp(2\mu\Delta_n) - 4\exp(\mu\Delta_n) + 1 \\ R_4 &= \exp((4\mu + \sigma^2)\Delta_n) - 2\exp((3\mu + \sigma^2)\Delta_n) + 3\exp((2\mu + \sigma^2)\Delta_n) \\ &\quad -2\exp(3\mu\Delta_n) + 5\exp(2\mu\Delta_n) - 4\exp(\mu\Delta_n) + 1\end{aligned}$$

and

$$R_5 = (\exp(\mu\Delta_n) - 1)^4.$$

We have now the following theorem for the consistency of $\hat{\mu}_{EU}$ and $\hat{\sigma}^2_{EU}$.

Theorem 6

The Euler-Maruyama estimates $\hat{\mu}_{EU}$ and $\hat{\sigma}^2_{EU}$ of the parameters of Lognormal process are both consistent if and only if the condition D_3 or D_4 is satisfied. The order of consistency of $\hat{\mu}_{EU}$ and $\hat{\sigma}^2_{EU}$ is then $\min(\sqrt{n}, 1/\Delta_n)$ and $1/\Delta_n$ respectively.

We now understand that Euler-Maruyama approximation heavily affects the behavior of the estimate of parameters, although this approximation is established as a standard tool of approximating the Ito type stochastic difference equation. From the view point of approximation of a process, the sampling interval Δ_n should be chosen as small as possible, but it is not true for estimation of parameters. For example the order of consistency of the estimate $\hat{\beta}_{EU}$ of β of Ornstein-Uhlenbeck process becomes lower as is seen from Theorem 4 and the estimate of σ^2 is consistent in the limited cases. Similar things happen also for Lognormal process.

2.3 Bootstrap Estimate

As is seen in the previous section, Euler-Maruyama approximation affects the behavior of estimates in either case, Ornstein-Uhlenbeck or Lognormal process. Even if Δ_n is carefully chosen, there remain cases where the order of consistency is significantly less than that of the exact maximum likelihood estimate. In this section, we will show that bootstrap estimate attains the same order of consistency as that of the exact maximum likelihood estimate does. This result suggests that bootstrap estimate is quite promising even when the exact solution of stochastic differential equation is unknown.

Bootstrap is a method of computation with the help of random number generation. More precisely, the expectation $\mathrm{E}(T(X_1,\cdots,X_n))$ of a statistic $T(X_1,\cdots,X_n)$ is calculated or approximated through its mean with respect to random numbers;

$$\mathrm{E}^*(T(X_1^*,\cdots,X_n^*)|X_1=x_1,\cdots,X_n=x_n), \tag{2.10}$$

where $(X_1^*,\cdots,X_n^*)$ stands for random numbers which simulate observations $(X_1,\cdots,X_n)$, and E^* denotes a conditional expectation with respect to $(X_1^*,\cdots,X_n^*)$ given $(X_1=x_1,\cdots,X_n=x_n)$. Practically the bootstrap expectation in (2.10) is replaced by an average

$$\frac{1}{B}\sum_{j=1}^{B} T(x_1^{*(j)},\cdots,x_n^{*(j)}), \tag{2.11}$$

where $(x_1^{*(j)},\cdots,x_n^{*(j)})$, $j=1,\cdots,B$ are sets of random numbers, that is, realizations of $(X_1^*,\cdots,X_n^*)$. The approximation error of (2.11) to (2.10) almost surely goes to zero as B tends to infinity because of law of large numbers. In many cases, approximation error of (2.10) to the expectation $\mathrm{E}(T(X_1,\cdots,X_n))$ also goes to zero as n tends to infinity, for example, in case of $X_1,\cdots,X_n$ being i.i.d..

There are several ways of generating random numbers. Most popular one is to generate random numbers based on empirical distribution of i.i.d. observations $x_1, \cdots, x_n$. Such a bootstrap is called *non-parametric.* Others are making use of models available in some way. For example, if a regression model is available for $X_1, \cdots, X_n$ then we first estimate regression parameters and generate $X_1^*, \cdots, X_n^*$ based on random numbers which follow empirical distribution of residuals. This type of bootstrap is called *semi-parametric.* If a specific distribution, for example, normal distribution, is employed in place of the empirical distribution of residuals, then such a method is called *parametric.* Thus, various type of bootstrap estimates are available. However in this paper we will concentrate our attention on parametric bootstrap estimate of likelihood function or of score function. Although how to choose the number B of bootstrap samples is a crucial problem in practice, we leave it for future investigation. We will discuss only (theoretical) bootstrap expectation in (2.10) and denote it simply as

$$\mathrm{E}^*(T(X_1^*, \cdots, X_n^*))$$

by omitting the conditions. In other words, we assume that infinitely many bootstrap samples are available and approximation error of (2.11) to (2.10) is negligible. We first consider bootstrap estimate of conditional density $f_{\boldsymbol{\theta}}(\cdot, |x_{t_{i-1}})$ of X_{t_i} given observations $x_{t_0}, \cdots, x_{t_{i-1}}$. Since we know that $\{X_t\}$ satisfies the stochastic differential equation (1.1), we can sequentially generate random numbers according to Euler-Maruyama approximation,

$$\begin{aligned} X^*_{t_{i-1,k}} &= X^*_{t_{i-1,k-1}} + a(X^*_{t_{i-1,k-1}} t_{i-1,k-1} : \boldsymbol{\theta})\Delta_n/N + b(X^*_{t_{i-1,k-1}} t_{i-1,k-1} : \boldsymbol{\theta})W_k, \\ & \quad k = 1, \cdots, N-1, \end{aligned} \tag{2.12}$$

where $t_{i-1} = t_{i-1,0} < t_{i-1,1} < t_{i-1,2} < \cdots, < t_{i-1,N-1} < t_i$ constitute a uniform partition of $[t_{i-1}, t_i)$. Variables $\{W_k, k = 1, \cdots, N-1\}$ denote independent normal random numbers with mean 0 and variance Δ_n/N.

Let us denote the conditional density of $X^*_{t_{i-1,k}}$ given $X^*_{t_{i-1,k-1}} = x^*_{t_{i-1,k-1}}$ by $f^{(1)}_{\boldsymbol{\theta}}(x^*_{t_{i-1,k}} | x^*_{t_{i-1,k-1}})$. The conditional density $f^{(N)}_{\boldsymbol{\theta}}(\cdot | x_{t_{i-1}})$ of the bootstrap for x_{t_i},

$$X^*_{t_i} = X^*_{t_{i-1,N-1}} + a(X^*_{t_{i-1,N-1}} t_{i-1,N-1} : \boldsymbol{\theta})\Delta_n/N + b(X^*_{t_{i-1,N-1}} t_{i-1,N-1} : \boldsymbol{\theta})W_N$$

is then simplified because of the Markov property of generation mechanism

(2.12) of random numbers. In fact, Pedersen(1995) showed that the equation

$$
\begin{aligned}
f_{\boldsymbol{\theta}}^{(N)}(x|x_{t_{i-1}}) &= \int \cdots \int \prod_{k=1}^{N-1} f_{\boldsymbol{\theta}}^{(1)}(x^*_{t_{i-1,k}}|x^*_{t_{i-1,k-1}}) f_{\boldsymbol{\theta}}^{(1)}(x|x^*_{t_{i-1,N-1}}) dx^*_{t_{i-1,1}} \cdots dx^*_{t_{i-1,N-1}} \\
&= \mathrm{E}^* \left\{ f_{\boldsymbol{\theta}}^{(1)}(x|x^*_{t_{i-1,N-1}}) \right\} \qquad (2.13)
\end{aligned}
$$

holds true and he also proved that $f_{\boldsymbol{\theta}}^{(N)}(x|x_{t_{i-1}})$ converges to $f_{\boldsymbol{\theta}}(x|x_{t_{i-1}})$ as N tends to infinity.

However, to obtain the maximum likelihood estimate of parameter it is necessary to estimate the score function in place of the log likelihood function. The following theorem gives us a practical method of constructing a bootstrap estimate of the score function.

Theorem 7

The score function based on observations, $x_{t_0}, x_{t_1}, \cdots, x_{t_n}$ is written as

$$
\begin{aligned}
&\frac{\partial}{\partial \boldsymbol{\theta}} l_n^N(\boldsymbol{\theta}) \\
&= \sum_{i=1}^{n} \Bigg[\mathrm{E}^* \Bigg\{ f_{\boldsymbol{\theta}}^{(1)}(x_{t_i}|x^*_{t_{i-1,N-1}}) \Bigg(\sum_{k=1}^{N-1} \frac{\partial}{\partial \boldsymbol{\theta}} \log f_{\boldsymbol{\theta}}^{(1)}(x^*_{t_{i-1,k}}|x^*_{t_{i-1,k-1}}) \\
&\quad + \frac{\partial}{\partial \boldsymbol{\theta}} \log f_{\boldsymbol{\theta}}^{(1)}(x_{t_i}|x^*_{t_{i-1,N-1}}) \Bigg) \Bigg\} \Bigg/ \mathrm{E}^* \left\{ f_{\boldsymbol{\theta}}^{(1)}(x_{t_i}|x^*_{t_{i-1,N-1}}) \right\} \Bigg] \qquad (2.14)
\end{aligned}
$$

for $N \geq 2$.

Proof

The score function can be written as

$$
\frac{\partial}{\partial \boldsymbol{\theta}} l_n^{(N)}(\boldsymbol{\theta}) = \sum_{i=1}^{n} \frac{\partial}{\partial \boldsymbol{\theta}} \log f_{\boldsymbol{\theta}}^{(N)}(x_{t_i}|x_{t_{i-1}}) = \sum_{i=1}^{n} \frac{\frac{\partial}{\partial \boldsymbol{\theta}} f_{\boldsymbol{\theta}}^{(N)}(x_{t_i}|x_{t_{i-1}})}{f_{\boldsymbol{\theta}}^{(N)}(x_{t_i}|x_{t_{i-1}})}.
$$

Numerators on the right hand side of the equation above can be rewritten as

$$
\frac{\partial}{\partial \boldsymbol{\theta}} f_{\boldsymbol{\theta}}^{(N)}(x_{t_i}|x_{t_{i-1}})
$$

$$
\begin{aligned}
&= \frac{\partial}{\partial \boldsymbol{\theta}} \int \prod_{k=1}^{N-1} f_{\boldsymbol{\theta}}^{(1)}(x^*_{t_{i-1,k}} | x^*_{t_{i-1,k-1}}) f_{\boldsymbol{\theta}}^{(1)}(x_{t_i} | x^*_{t_{i-1,N-1}}) dx^*_{t_{i-1,1}} \cdots dx^*_{t_{i-1,N-1}} \\
&= \int \Big\{ f_{\boldsymbol{\theta}}^{(1)}(x_{t_i} | x^*_{t_{i-1,N-1}}) \frac{\partial}{\partial \boldsymbol{\theta}} \prod_{k=1}^{N-1} f_{\boldsymbol{\theta}}^{(1)}(x^*_{t_{i-1,k}} | x^*_{t_{i-1,k-1}}) \\
&\quad + \prod_{k=1}^{N-1} f_{\boldsymbol{\theta}}^{(1)}(x^*_{t_{i-1,k}} | x^*_{t_{i-1,k-1}}) \frac{\partial}{\partial \boldsymbol{\theta}} f_{\boldsymbol{\theta}}^{(1)}(x_{t_i} | x^*_{t_{i-1,N-1}}) \Big\} dx^*_{t_{i-1,1}} \cdots dx^*_{t_{i-1,N-1}} \\
&= \int \Big\{ f_{\boldsymbol{\theta}}^{(1)}(x_{t_i} | x^*_{t_{i-1,N-1}}) \frac{\partial}{\partial \boldsymbol{\theta}} \prod_{k=1}^{N-1} f_{\boldsymbol{\theta}}^{(1)}(x^*_{t_{i-1,k}} | x^*_{t_{i-1,k-1}}) \Big/ \prod_{k=1}^{N-1} f_{\boldsymbol{\theta}}^{(1)}(x^*_{t_{i-1,k}} | x^*_{t_{i-1,k-1}}) \\
&\quad + \frac{\partial}{\partial \boldsymbol{\theta}} f_{\boldsymbol{\theta}}^{(1)}(x_{t_i} | x^*_{t_{i-1,N-1}}) \Big\} \prod_{k=1}^{N-1} f_{\boldsymbol{\theta}}^{(1)}(x^*_{t_{i-1,k}} | x^*_{t_{i-1,k-1}}) dx^*_{t_{i-1,1}} \cdots dx^*_{t_{i-1,N-1}} \\
&= \mathrm{E}^* \left\{ f_{\boldsymbol{\theta}}^{(1)}(x_{t_i} | x^*_{t_{i-1,N-1}}) \left(\sum_{k=1}^{N-1} \frac{\partial}{\partial \boldsymbol{\theta}} \log f_{\boldsymbol{\theta}}^{(1)}(x^*_{t_{i-1,k}} | x^*_{t_{i-1,k-1}}) + \frac{\partial}{\partial \boldsymbol{\theta}} \log f_{\boldsymbol{\theta}}^{(1)}(x_{t_i} | x^*_{t_{i-1,N-1}}) \right) \right\}.
\end{aligned}
$$

Then, it is enough to note the equation (2.13) for denominators to prove Theorem 7.

A practical bootstrap estimate of score function is obtained by replacing bootstrap expectations in numerators and denominators on the right hand side of (2.14) in Theorem 7 by averages with respect to B times random number generations. Of course, to obtain actual parameter estimate it is necessary to do a search for zeros of the score function. In due course, various problems may arise in practice, for example, choice of initial value of $\boldsymbol{\theta}$, stopping rule for the search, efficient generation of bootstrap samples and so on. We don't discuss such practical problems in this paper. Instead, we only evaluate order of consistency of $\hat{\boldsymbol{\theta}}_{BS}$ which is the solution of

$$
\frac{\partial}{\partial \boldsymbol{\theta}} l_n^N(\boldsymbol{\theta}) = 0. \tag{2.15}
$$

2.3.1 Ornstein-Uhlenbeck Process

To obtain the solution of (2.15), noting that

$$
f_{\boldsymbol{\theta}}^{(1)}(x_{t_i} | x^*_{t_{i-1,N-1}}) = \frac{1}{(2\pi \Delta_n / N)^{\frac{1}{2}} \sigma} \exp\left\{ - \frac{(x_{t_i} - (1 - \beta \Delta_n / N) x^*_{t_{i-1,N-1}})^2}{2\sigma^2 \Delta_n / N} \right\}
$$

we have

$$
\begin{aligned}
& f_{\boldsymbol{\theta}}^{(N)}(x_{t_i}|x_{t_{i-1}}) \\
&= \mathrm{E}^*\left\{f_{\boldsymbol{\theta}}^{(1)}(x_{t_i}|X^*_{t_{i-1},N-1})\right\} \\
&= \mathrm{E}^*\left[\frac{1}{(2\pi\Delta_n/N)^{\frac{1}{2}}\sigma}\exp\left\{-\frac{(x_{t_i}-(1-\beta\Delta_n/N)X^*_{t_{i-1},N-1})^2}{2\sigma^2\Delta_n/N}\right\}\right].
\end{aligned}
$$

Since the conditional distribution of $X^*_{t_{i-1},N-1}$ given observations x_{t_i-1} is normal with the mean $(1-\beta\Delta_n/N)^{N-1}x_{t_{i-1}}$ and the variance $\sigma^2\sum_{k=0}^{N-2}(1-\beta\Delta_n/N)^{2k}\Delta_n/N$ from the generation mechanism (2.12) of random numbers, the bootstrap expectation above is a convolution of two normal distributions. Therefore it can be simplified as

$$
f_{\boldsymbol{\theta}}^{(N)}(x_{t_i}|x_{t_{i-1}}) = \frac{1}{(2\pi\eta_N)^{\frac{1}{2}}}\exp\left\{-\frac{\left(x_{t_i}-(1-\beta\Delta_n/N)^N x_{t_{i-1}})\right)^2}{2\eta_N}\right\},
$$

where

$$
\begin{aligned}
\eta_N &= \sigma^2\frac{\Delta_n}{N}\left(1+\sum_{k=0}^{N-2}(1-\beta\Delta_n/N)^{2k+2}\right) \\
&= \sigma^2\frac{\Delta_n}{N}\frac{1-(1-\beta\Delta_n/N)^{2N}}{1-(1-\beta\Delta_n/N)^2}.
\end{aligned}
$$

As a result, the bootstrap estimate of score function with respect to β becomes

$$
\frac{\partial}{\partial\beta}l_n^N(\boldsymbol{\theta}) = -\frac{1}{\eta_N}\sum_{i=1}^{n}\left\{\left(x_{t_i}-(1-\beta\Delta_n/N)^N x_{t_{i-1}}\right)(1-\beta\Delta_n/N)^{N-1}x_{t_{i-1}}\Delta_n\right\}
$$

and the estimate $\hat{\beta}_{BS}$ is the solution of

$$
(1-\beta\Delta_n/N)^N = \frac{\sum_{i=1}^n x_{t_i}x_{t_{i-1}}}{\sum_{i=1}^n x_{t_i}^2}.
$$

On the other hand, the bootstrap estimate of η_N becomes

$$
\begin{aligned}
\hat{\eta}_{BS} &= \frac{1}{n}\sum_{i=1}^{n}\left\{x_{t_i}-(1-\hat{\beta}_{BS}\Delta_n/N)^N x_{t_{i-1}}\right\}^2 \\
&= \hat{\gamma}_{MLE}.
\end{aligned}
$$

Therefore, the bootstrap estimate of σ^2 is written as

$$\hat{\sigma}^2_{BS} = \frac{N}{\Delta_n} \frac{1-(1-\hat{\beta}_{BS}\Delta_n/N)^2}{1-(1-\hat{\beta}_{BS}\Delta_n/N)^{2N}} \hat{\gamma}_{MLE}.$$

The reader may feel strange that both the score function and all estimates involve no bootstrap samples. This is because we assumed that infinitely many bootstrap samples are available. In practice, the score function or the estimate is more complicated. It is, however, enough to investigate behavior of estimates $\hat{\beta}_{BS}$ and $\hat{\sigma}^2_{BS}$ for the aim of this paper.

Theorem 8

The estimate $\hat{\beta}_{BS}$ is consistent in one of the following cases.

Case 1. $\beta > 0$

If Δ_n converges to 0 or Δ_n is bounded but $n\Delta_n$ diverges to infinity, then $\hat{\beta}_{BS}$ is consistent as far as N/Δ_n diverges to infinity. The order of consistency is $\min((n\Delta_n)^{\frac{1}{2}}, N/\Delta_n)$. It is also consistent if $\sqrt{n}d_n\Delta_n$ and N/Δ_n simultaneously diverge to infinity. The order of consistency is then $\min(\sqrt{n}d_n\Delta_n, N/\Delta_n)$.

Case 2. $\beta = 0$

If $n\Delta_n$ diverges to infinity, then $\hat{\beta}_{BS}$ is consistent. The order of consistency is $n\Delta_n$ if x_{t_0} is nonzero, otherwise $(n\Delta_n)^{\frac{3}{2}}$.

Case 3. $\beta < 0$

If $n\Delta_n$ diverges to infinity, $\hat{\beta}_{BS}$ is consistent as far as N/Δ_n diverges to infinity. The order of consistency is $\min(\Delta_n d_n^{n-1}, N/\Delta_n)$ if Δ_n diverges to infinity and $\min((\Delta_n/n)^{\frac{1}{2}} d_n^{2(n-1)}, N/\Delta_n)$ otherwise.

Proof

The results follow from Table 3 if we note that

$$\hat{\beta}_{BS} - \beta = \frac{N}{\Delta_n}\left\{1 - \beta\Delta_n/N - \exp(-\beta\Delta_n/N)\left(\frac{\xi_n}{\eta_n}+1\right)^{\frac{1}{N}}\right\}.$$

2.3.2 Lognormal Process

We consider first bootstrap estimate of parameters of a transformed process $Y_t = \log X_t$ which satisfies

$$dY_t = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dB_t.$$

Then

$$f^{(1)}_{\boldsymbol{\theta}}(y_{t_i}|y^*_{t_{i-1,N-1}}) = \frac{1}{(2\pi\Delta_n/N)^{\frac{1}{2}}\sigma} \exp\left\{-\frac{(y_{t_i} - y^*_{t_{i-1,N-1}} - \alpha\frac{\Delta_n}{N})^2}{2\sigma^2\Delta_n/N}\right\} \quad (2.16)$$

for $\boldsymbol{\theta} = (\alpha, \sigma^2)$ with $\alpha = \mu - (1/2)\sigma^2$. The bootstrap expectation of (2.16) is easily calculated by using the moment generating function of non-central χ^2 distribution and

$$\mathrm{E}^*\left\{f^{(1)}_{\boldsymbol{\theta}}(y_{t_i}|y^*_{t_{i-1,N-1}})\right\} = \frac{1}{(2\pi\Delta_n)^{\frac{1}{2}}\sigma} \exp\left\{-\frac{(y_{t_i} - y_{t_{i-1}} - \alpha\Delta_n)^2}{2\sigma^2\Delta_n}\right\}.$$

Therefore the score functions become

$$\frac{\partial}{\partial\alpha} l^N_n(\boldsymbol{\theta}) = \sum_{i=1}^{n}(y_{t_i} - y_{t_{i-1}} - \alpha\Delta_n)/\sigma^2$$

and

$$\frac{\partial}{\partial\sigma^2} l^N_n(\boldsymbol{\theta}) = \sum_{i=1}^{n}\left\{-\frac{(y_{t_i} - y_{t_{i-1}} - \alpha\Delta_n)^2}{2\sigma^4\Delta_n} - \frac{1}{2\sigma^2}\right\}.$$

We see that the solution coincides with the exact maximum likelihood estimate given in Section 2.1.2.

3 Summary of the Results

We have investigated consistency of three types of estimates for two types of processes. In this section we will summarize all results according to the type of processes.

3.1 Ornstein-Uhlenbeck process

The order of consistency is summarized in the following tables, where

$$
\begin{aligned}
\alpha_n &= \Delta_n / \log(1 + 1/(\sqrt{n} d_n)), \quad \beta_n = \begin{cases} n\Delta_n & \text{if } x_{t_0} = 0 \\ (n\Delta_n)^{\frac{3}{2}} & \text{otherwise} \end{cases}, \\
\varepsilon_n &= \begin{cases} \sqrt{n}\Delta_n & \text{if } \sqrt{n}\Delta_n \to \infty \\ \min(\sqrt{n}, \alpha_n) & \text{otherwise} \end{cases} \\
\text{and} \quad d_n &= \exp(-\beta\Delta_n).
\end{aligned}
$$

In the following tables, as is defined in Section 2 D_1 to D_4 are the cases where $\Delta_n \to \infty$, Δ_n is bounded and bounded away from 0, $\Delta_n \to 0$ but $n\Delta_n \to \infty$ and $n\Delta_n$ is bounded, respectively. Note that, even if the order of consistency is given in the table, there are cases where the entry does not give us any divergent sequence so that the estimate is not consistent for a particular choice of Δ_n and N.

Table 4. Order of Consistency of Estimates of β.

Δ_n	Estimate	$\beta > 0$	$\beta = 0$	$\beta < 0$
D_1	$\hat{\beta}_{MLE}$	α_n	β_n	$\Delta_n d_n^{n-1}$
	$\hat{\beta}_{EU}$	inconsistent	β_n	inconsistent
	$\hat{\beta}_{BS}$	$\min(\sqrt{n}\Delta_n d_n, N/\Delta_n)$	β_n	$\min(\Delta_n d_n^{n-1}, N/\Delta_n)$
D_2	$\hat{\beta}_{MLE}$	$\sqrt{n}$	β_n	$n^{-\frac{1}{2}} d_n^{2(n-1)}$
	$\hat{\beta}_{EU}$	inconsistent	β_n	inconsistent
	$\hat{\beta}_{BS}$	$\min(\sqrt{n}, N)$	β_n	$\min(n^{-\frac{1}{2}} d_n^{2(n-1)}, N)$
D_3	$\hat{\beta}_{MLE}$	$(n\Delta_n)^{\frac{1}{2}}$	β_n	$(\Delta_n/n)^{\frac{1}{2}} d_n^{2(n-1)}$
	$\hat{\beta}_{EU}$	$(n\Delta_n)^{\frac{1}{2}}$	β_n	$\Delta_n d_n^{n-1}$
	$\hat{\beta}_{BS}$	$\min((n\Delta_n)^{\frac{1}{2}}, N/\Delta_n)$	β_n	$\min((\Delta_n/n)^{\frac{1}{2}} d_n^{2(n-1)}, N/\Delta_n)$
D_4	$\hat{\beta}_{MLE}$	inconsistent	inconsistent	inconsistent
	$\hat{\beta}_{EU}$	inconsistent	inconsistent	inconsistent
	$\hat{\beta}_{BS}$	inconsistent	inconsistent	inconsistent

Table 5. Order of Consistency of Estimates of σ^2 when $t_0 \neq 0$.

Δ_n	Estimate	$\beta > 0$	$\beta = 0$	$\beta < 0$
D_1	$\hat{\sigma}^2_{MLE}$	ε_n	$\sqrt{n}$	$\min(\sqrt{n}, d_n^{n-1})$
	$\hat{\sigma}^2_{EU}$	inconsistent	$\sqrt{n}$	inconsistent
	$\hat{\sigma}^2_{BS}$	inconsistent	$\min(\sqrt{n}\Delta_n, N)$	inconsistent
D_2	$\hat{\sigma}^2_{MLE}$	$\sqrt{n}$	$\min(\sqrt{n}, \beta_n)$	$\min(\sqrt{n}, d_n^{2(n-1)}/\sqrt{n})$
	$\hat{\sigma}^2_{EU}$	inconsistent	$\sqrt{n}$	inconsistent
	$\hat{\sigma}^2_{BS}$	inconsistent	$\min(\sqrt{n}\Delta_n, N)$	inconsistent
D_3	$\hat{\sigma}^2_{MLE}$	$(n\Delta_n)^{\frac{1}{2}}$	$\min(\sqrt{n}, \beta_n)$	$\min(\sqrt{n}, (\Delta_n/n)^{\frac{1}{2}} d_n^{2(n-1)})$
	$\hat{\sigma}^2_{EU}$	inconsistent	$\sqrt{n}$	inconsistent
	$\hat{\sigma}^2_{BS}$	inconsistent	$\min(\sqrt{n}\Delta_n, N)$	inconsistent
D_4	$\hat{\sigma}^2_{MLE}$	inconsistent	inconsistent	inconsistent
	$\hat{\sigma}^2_{EU}$	inconsistent	$\sqrt{n}$	inconsistent
	$\hat{\sigma}^2_{BS}$	inconsistent	inconsistent	inconsistent

Table 6. **Order of Consistency of Estimates of σ^2 when $t_0 = 0$.**

Δ_n	Estimate	$\beta > 0$	$\beta = 0$	$\beta < 0$
D_1	$\hat{\sigma}^2_{MLE}$ $\hat{\sigma}^2_{EU}$ $\hat{\sigma}^2_{BS}$	ε_n inconsistent inconsistent	$\sqrt{n}$ $\sqrt{n}$ inconsistent	$\min(\sqrt{n}, d_n^{n-1})$ inconsistent inconsistent
D_2	$\hat{\sigma}^2_{MLE}$ $\hat{\sigma}^2_{EU}$ $\hat{\sigma}^2_{BS}$	$\sqrt{n}$ inconsistent inconsistent	$\sqrt{n}$ $\sqrt{n}$ inconsistent	$\min(\sqrt{n}, d_n^{2(n-1)}/\sqrt{n})$ inconsistent inconsistent
D_3	$\hat{\sigma}^2_{MLE}$ $\hat{\sigma}^2_{EU}$ $\hat{\sigma}^2_{BS}$	$(n\Delta_n)^{\frac{1}{2}}$ $\min(\sqrt{n}, 1/\Delta_n)$ $\min(1/\Delta_n, \sqrt{n}\Delta_n, N)$	$\sqrt{n}$ $\sqrt{n}$ $\sqrt{n}\Delta_n$	$\min(\sqrt{n}, (\Delta_n/n)d_n^{2(n-1)})$ $\min(\sqrt{n}, 1/\Delta_n)$ $\min(1/\Delta_n, d_n^{2(n-1)}/\sqrt{n\Delta_n}, N)$
D_4	$\hat{\sigma}^2_{MLE}$ $\hat{\sigma}^2_{EU}$ $\hat{\sigma}^2_{BS}$	inconsistent $\min(\sqrt{n}, 1/\Delta_n)$ inconsistent	inconsistent $\sqrt{n}$ inconsistent	inconsistent $\min(\sqrt{n}, 1/\Delta_n)$ inconsistent

3.2 Lognormal Process

The order of consistency in case of lognormal process is simple. It does not depend on the value of parameters. The results are summarized in Table 7.

Table 7. Order of Consistency of Estimates of μ and σ^2.

Δ_n	Estimate		Estimate	
$D_1 \vee D_2$	$\hat{\mu}_{MLE}$ $\hat{\mu}_{EU}$ $\hat{\mu}_{BS}$	$\sqrt{n}$ inconsistent $\sqrt{n}$	$\hat{\sigma}^2_{MLE}$ $\hat{\sigma}^2_{EU}$ $\hat{\sigma}^2_{BS}$	$\sqrt{n}$ inconsistent $\sqrt{n}$
$D_3 \vee D_4$	$\hat{\mu}_{MLE}$ $\hat{\mu}_{EU}$ $\hat{\mu}_{BS}$	$(n\Delta_n)^{\frac{1}{2}}$ $\min(\sqrt{n}, 1/\Delta_n)$ $(n\Delta_n)^{\frac{1}{2}}$	$\hat{\sigma}^2_{MLE}$ $\hat{\sigma}^2_{EU}$ $\hat{\sigma}^2_{BS}$	$\sqrt{n}$ $1/\Delta_n$ $\sqrt{n}$

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